1.

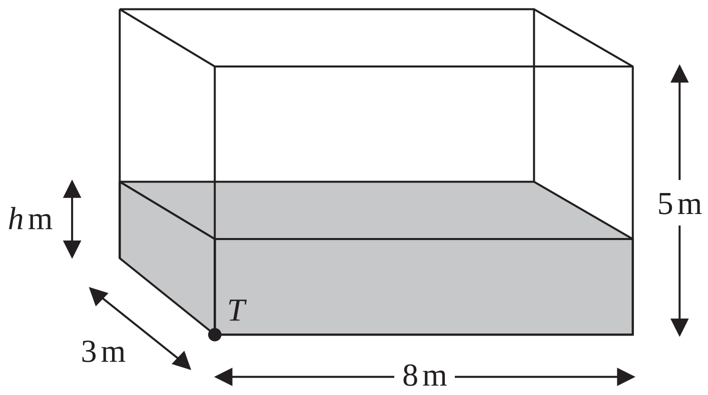

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

a)

let V = volume of water in the tank at time t

$V = 3 \times 8 \times h = 24h$

$\frac{dV}{dh} = 24$ 1

given that water moves in at $0.48 m^3$ per minute and water moves out at $0.1h \, m^3$ per minute

$\therefore \frac{dV}{dt} = 0.48 - 0.1h$ 1

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{24} \times (0.48 - 0.1h)$ 1

$\frac{dh}{dt} = \frac{24}{1200} - \frac{5h}{1200}$

$1200 \frac{dh}{dt} = 24 - 5h$ 1

b) when $t = 0$, $h = 2$

$1200 \frac{dh}{dt} = 24 - 5h$ separate variables

$\Rightarrow \int \frac{1200}{24 - 5h} \, dh = \int dt$ 1

if $y = \ln(24 - 5h)$

$\frac{dy}{dh} = \frac{-5}{24 - 5h}$

$-240 \ln|24 - 5h| = t + c$ 1

we need numerator = 1200, so multiply by −240

$-240\ln|24-5h| = t + c$

sub in $t=0, h=2$

$0 + c = -240\ln|24-10|$
$c = -240\ln 14$ (1)

$\therefore \quad -240\ln|24-5h| = t - 240\ln 14$

$t = 240\ln 14 - 240\ln|24-5h|$ (1)

$$\frac{t}{240} = \ln\left(\frac{14}{24-5h}\right)$$

$$\Rightarrow e^{\frac{t}{240}} = \frac{14}{24-5h}$$

$$24e^{\frac{t}{240}} - 5he^{\frac{t}{240}} = 14$$

$$5he^{\frac{t}{240}} = 24e^{\frac{t}{240}} - 14 \quad (1)$$

$$h = \frac{24e^{t/240} - 14}{5e^{t/240}}$$

$$h = 4.8 - 2.8e^{-\frac{t}{240}} \quad (1)$$

c) as $t \to \infty$, $e^{-\frac{t}{240}} \to 0$ so $h \to 4.8$ (1)

The tank is 5m high, and the limit for h is 4.8m, (1)
so the tank will never become full.

2. A scientist is studying the number of bees and the number of wasps on an island.

The number of bees, measured in thousands, N_b, is modelled by the equation

$$N_b = 45 + 220\,e^{0.05t}$$

where t is the number of years from the start of the study.

According to the model,

(a) find the number of bees at the start of the study,

(1)

(b) show that, exactly 10 years after the start of the study, the number of bees was increasing at a **rate** of approximately 18 thousand per year.

(3)

The number of wasps, measured in thousands, N_w, is modelled by the equation

$$N_w = 10 + 800\,e^{-0.05t}$$

where t is the number of years from the start of the study.

When $t = T$, according to the models, there are an equal number of bees and wasps.

(c) Find the value of T to 2 decimal places.

(4)

(a) when $t = 0$:

$N_b = 45 + 220e^{0.05 \times 0}$

$= 45 + 220e^{0}$ ← $e^0 = 1$

$= 45 + 220$

$= 265$

265 thousand (1)

(b) $\frac{dN_b}{dt} = 0.05 \times 220 \times e^{0.05t}$ ← differentiate w.r.t time to get rate of change.

$= 11e^{0.05t}$ (1)

$\frac{d}{dx} e^{ax} = ae^{ax}$

when $t = 10$.

$\frac{dN_b}{dt} = 11e^{0.05 \times 10}$ (1)

$= 18.135...$

which is approximately 18 thousand bees per year (1)

(c) when $t = T$, $N_b = N_w$:

$$45 + 220e^{0.05t} = 10 + 800e^{-0.05t}$$ (1)

$-10, -800e^{-0.05t}$

$$220e^{0.05t} + 35 - 800e^{-0.05t} = 0$$

$\times e^{0.05t}$

(1) $$220(e^{0.05t})^2 + 35e^{0.05t} - 800 = 0$$

Do this to remove the $e^{-0.05t}$ term.

$e^{0.05t} \times e^{-0.05t} = e^0 = 1$

This is a quadratic $220x^2 + 35x - 800 = 0$ with $x = e^{0.05t}$. Solve with calculator.

$$e^{0.05t} = 1.829, -1.988$$

ignore negative result because e^n cannot be negative

$$0.05t = \ln(1.829)$$ (1)

$$t = 12.08 \quad (2dp)$$

$\therefore T = 12.08$ years (1)

3.

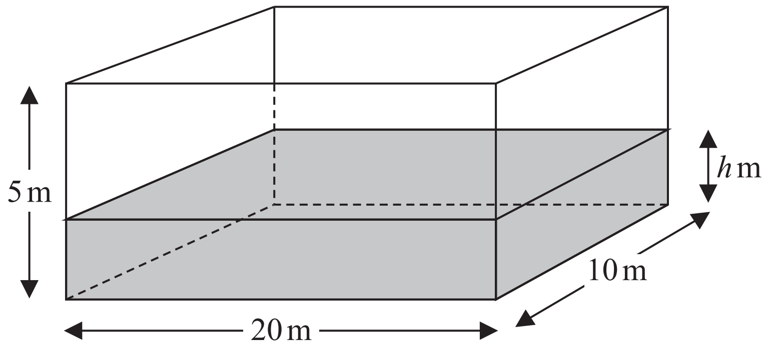

Figure 1

A tank in the shape of a cuboid is being filled with water.

The base of the tank measures 20 m by 10 m and the height of the tank is 5 m, as shown in Figure 1.

At time t minutes after water started flowing into the tank the height of the water was h m and the volume of water in the tank was V m^3

In a model of this situation

- the sides of the tank have negligible thickness
- the rate of change of V is inversely proportional to the square root of h

(a) Show that

$$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$$

where λ is a constant.

(3)

Given that

- initially the height of the water in the tank was 1.44 m
- exactly 8 minutes after water started flowing into the tank the height of the water was 3.24 m

(b) use the model to find an equation linking h with t, giving your answer in the form

$$h^{\frac{3}{2}} = At + B$$

where A and B are constants to be found.

(5)

(c) Hence find the time taken, from when water started flowing into the tank, for the tank to be completely full.

(2)

a) $V = 20 \times 10 \times h$ $\qquad \frac{dV}{dt} = \frac{k}{\sqrt{h}}$

$V = 200h$

$\frac{dV}{dh} = 200$ (1)

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$$ (1) (1) (where $\lambda = \frac{k}{200}$)

b) when $t=0$, $h=1.44$

when $t=8$, $h=3.24$

$$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}} \Rightarrow \int h^{1/2}\,dh = \int \lambda\,dt$$ (1)

$$\frac{2}{3}h^{3/2} = \lambda t + c$$ (1)

sub in $t=0$, $h=1.44$

$$\frac{2}{3}(1.44)^{3/2} = 0\lambda + c \Rightarrow c = 1.152$$ (1)

sub in $t=8$, $h=3.24$

$$\frac{2}{3}(3.24)^{3/2} = 8\lambda + 1.152 \Rightarrow \lambda = 0.342$$ (1)

$$\frac{2}{3}h^{3/2} = 0.342t + 1.152$$

$$h^{3/2} = 0.513t + 1.728$$ (1)

c) tank is full when $h=5$

$$(5)^{3/2} = 0.513t + 1.728$$ (1)

$$t = \frac{5\sqrt{5} - 1.728}{0.513} \Rightarrow t = 18.4 \text{ minutes } (3sf)$$ (1)